

Multiple phase estimation for arbitrary pure states under white noise

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(Dated: September 9, 2014)

In any realistic quantum metrology scenarios, the ultimate precision in the estimation of parameters is limited not only by the so-called Heisenberg scaling, but also the environmental noise encountered by the underlying system. In the context of quantum estimation theory, it is of great significance to carefully evaluate the impact of a specific type of noise on the corresponding quantum Fisher information (QFI) or quantum Fisher information matrix (QFIM). Here we investigate the *multiple* phase estimation problem for a natural parametrization of arbitrary pure states under *white noise*. We obtain the explicit expression of the symmetric logarithmic derivative (SLD) and hence the analytical formula of QFIM. Moreover, the attainability of the quantum Cramér-Rao bound (QCRB) is confirmed by the commutability of SLDs and the optimal estimators are elucidated for the experimental purpose. These findings generalize previously known partial results and highlight the role of white noise in quantum metrology.

PACS numbers: 03.65.Yz, 06.20.-f

Quantum metrology, emerged as a new branch of quantum technologies, provides a powerful and versatile framework for both theoretical and experimental studies in the field of quantum-enhanced parameter estimation [1–4]. However, any realistic physical system will suffer from various environmental noises via the coupling with its surroundings [5]. As pointed out in Ref. [4], analysis of the effects of noise is one of the major burgeoning trends of this field. With the efforts of multiple authors, it is clearly evident that even a very low noise level can destroy the quadratic improvement over the classical shot-noise limit [7–10]. Although a unified method to deal with noise of arbitrary form is still lacking, more in-depth study in this respect is continuing and the scope is far beyond the usual noisy quantum channels raised in [9, 10]. In fact, a plenty variety of significant physical effects or processes can also be regarded as the corresponding noisy quantum channels in the context of quantum information theory. For instance, quite recently it is demonstrated that the relativistic effect and quantum cloning machines are excellent platforms for investigating the quantum feature of quantum metrology scenarios [11–16].

On the other hand, due to the quantum Cramér-Rao inequality, quantum Fisher information (QFI) is recognized as the key quantity to characterize the ultimate precision in parameter estimation scenarios [17–20]. Therefore, a great amount of research work of noisy quantum metrology can be translated into the evaluation of the dynamics of QFI in the presence of a certain kind of noise. Though different kinds of upper bounds on QFI

have been obtained for various purposes [9, 10, 21–25], the analytical treatment of QFI is usually a difficult task. To summarize, we realize that all these analytical approaches in the literature can be classified into the following three categories.

Method I. As shown by the seminal work of Braunstein and Caves, the QFI is intimately related to the the Bures distance or Uhlmann fidelity [19]

$$\begin{aligned}\mathcal{F}(\theta) &= 4 \lim_{\epsilon \rightarrow 0} \left[\frac{\partial d_B(\rho_\theta, \rho_{\theta+\epsilon})}{\partial \epsilon} \right]^2 \\ &= -2 \lim_{\epsilon \rightarrow 0} \frac{\partial^2 F_U(\rho_\theta, \rho_{\theta+\epsilon})}{\partial \epsilon^2},\end{aligned}\quad (1)$$

where the Bures distance (and Uhlmann fidelity) between two quantum states ρ_1, ρ_2 can be defined as [26–28]

$$\begin{aligned}d_B(\rho_1, \rho_2) &= \sqrt{2 - 2\sqrt{F_U(\rho_1, \rho_2)}}, \\ F_U(\rho_1, \rho_2) &= \left(\text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right)^2.\end{aligned}\quad (2)$$

Therefore, instead of direct derivation, we can exploit the above relation to access the analytical formula of QFI if one has already obtained the explicit expression of the Bures distance (or Uhlmann fidelity) of the corresponding states. Actually, we recently notice that this strategy has already been applied successfully in several situations: the single qubit [29], single-mode Gaussian [30], and two-mode Gaussian states [12, 31].

Method II. This strategy is based on the the spectral decomposition of the density operator $\rho(\theta)$

$$\rho(\theta) = \sum_{k=1}^d \lambda_k(\theta) |\psi_k(\theta)\rangle \langle \psi_k(\theta)|, \quad (3)$$

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where d is the dimension of $\rho(\theta)$ and θ is the parameter to be estimated. Note that $\lambda_k(\theta)$ might be zero for some k . Using the Eq. (3) as the starting point, Pairs and O’Loan provided an explicit expression of QFI [3, 32]

$$\mathcal{F}_\theta = \sum_{i=1}^d \frac{1}{\lambda_i} (\partial_\theta \lambda_i)^2 + 2 \sum_{j \neq k}^d \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} |\langle \psi_j | \partial_\theta \psi_k \rangle|^2, \quad (4)$$

Consequently, Liu and Zhang *et al.* went a step further, by noting that the symmetric logarithmic derivative (SLD) is only defined on the support of $\rho(\theta)$. Therefore, the QFI can be rewritten as [33–35]

$$\begin{aligned} \mathcal{F}_\theta = & \sum_{i=1}^r \frac{1}{\lambda_i} (\partial_\theta \lambda_i)^2 + \sum_{i=1}^r 4\lambda_i \mathcal{F}_{Q,i} \\ & - \sum_{j \neq k}^r \frac{8\lambda_j \lambda_k}{\lambda_j + \lambda_k} |\langle \psi_j | \partial_\theta \psi_k \rangle|^2, \end{aligned} \quad (5)$$

where the QFI for pure eigenstate reads

$$\mathcal{F}_{Q,i} = 4 \left(\langle \partial_\theta \psi_i | \partial_\theta \psi_i \rangle - |\langle \psi_i | \partial_\theta \psi_i \rangle|^2 \right). \quad (6)$$

It should be emphasizing that now the summations go over the set $1 \leq k \leq r$ and r is the rank of $\rho(\theta)$. The above expression is more convenient for the non-full-rank states and gives a clear physical meaning [33–35]. However, for arbitrarily high-dimensional states, it is not so easy to obtain a compact decomposition basis, especially when the degeneracy of eigenvalues emerges [16]. Numerical studies may benefit more from this formula.

Method III. The QFI is defined in terms of the SLD which satisfies the following the equation

$$\frac{\partial \rho_\theta}{\partial \theta} = \frac{1}{2} (\rho_\theta \mathcal{L}_\theta + \mathcal{L}_\theta \rho_\theta). \quad (7)$$

If we obtain the explicit form of the Hermitian operator \mathcal{L}_θ , then the calculation of QFI will be an easy task. Nevertheless, the derivation of \mathcal{L}_θ is highly dependent on the structure of the density operator $\rho(\theta)$ and its parametrization. For several special cases, the analytical solution of \mathcal{L}_θ can be found, involving some mathematical tricks [36–39].

In this work, we thoroughly investigate the *multiple* phase estimation problem for a natural parametrization of arbitrary pure states under *white noise*. The effect of white noise, also known as the (isotropic) depolarizing channel [5] or *Werner* state [40], is given by the map

$$|\Psi\rangle\langle\Psi| \Rightarrow \rho^w = \eta |\Psi\rangle\langle\Psi| + \frac{1-\eta}{d} \mathbb{I}_{d \times d}. \quad (8)$$

where η is called the *reliability* of the channel. This form of states has already played an essential role in various quantum information tasks, such as quantum repeaters [41], NMR quantum computing [42] and quantum cloning machines [43]. To accurately formulate the current problem and compare with the previous results [16], we focus

on the following parametrization of arbitrary pure states

$$|\Psi(\phi)\rangle = \sum_{k=0}^{d-1} c_k e^{i\phi_k} |k\rangle. \quad (9)$$

Without loss of generality, c_k is assumed to be real since any imaginary part can be absorbed into the factor $e^{i\phi_k}$. Now the parameter vector $\phi = \{\phi_0, \phi_2, \dots, \phi_{d-1}\}$ is the target to be inferred. However, the overall phase cannot be estimated, so we can assume $\phi_0 = 0$. Here we adopt the *Method III*, that is, trying to find out the analytical expression of the SLDs for ρ^w .

Calculation of SLD. Before proceeding, the key observation which enables our calculation is that ρ^w can be represented in the exponential form

$$\rho^w = \eta \mathbb{P}(\phi) + \frac{1-\eta}{d} \mathbb{I}_{d \times d} = e^{\alpha \mathbb{P}(\phi) + \beta}, \quad (10)$$

by noting that

$$e^{\alpha \mathbb{P}(\phi)} = \mathbb{I} + (e^\alpha - 1) \mathbb{P}(\phi). \quad (11)$$

For simplicity, here we define the von Neumann-type projector $\mathbb{P}(\phi) = |\Psi(\phi)\rangle\langle\Psi(\phi)|$. Through direct calculation, we get the corresponding coefficients

$$\alpha = \ln \frac{(d-1)\eta + 1}{1-\eta}, \quad \beta = \ln \frac{1-\eta}{d}. \quad (12)$$

For states in the exponential form (e.g., $\rho(\theta) = e^{G(\theta)}$), Jiang provided a formal solution to the SLD [39]. The derivation is based on two main observations. First, the derivative of $\rho(\theta) = e^{G(\theta)}$ can be cast into an integral formula

$$\dot{\rho} = \int_0^1 e^{sG} \dot{G} e^{(1-s)G} ds. \quad (13)$$

where the overdot denotes the derivative with respect to θ . Secondly, utilizing the Baker-Hausdorff formula, we have

$$e^G A e^{-G} = \sum_{n=0}^{\infty} \frac{1}{n!} G^{\times n}(A) = e^{G^\times} A, \quad (14)$$

where the superoperator G^\times is introduced and G^\times denotes a commutator operation, namely [44]

$$G^\times(A) = [G, A] = GA - AG. \quad (15)$$

Combining Eqs. (7), (13) and (14), a formal expression of the SLD can be obtained [39]

$$\mathcal{L} = \sum_{n=0}^{\infty} f_n G^{\times n}(\dot{G}) = f(G^\times)(\dot{G}), \quad (16)$$

where the generating function f is determined by

$$f(t) = \sum_{n=0}^{\infty} f_n t^n = \frac{\tanh(t/2)}{t/2}, \quad (17)$$

To facilitate the solution of our problem, we define the following operator

$$\mathcal{A}_k = |\partial_{\phi_k} \Psi\rangle \langle \Psi| = ic_k e^{i\phi_k} |k\rangle \langle \Psi|, \quad (18)$$

Thus the derivative of \mathbb{P} with respect to ϕ_k is equal to

$$\dot{\mathbb{P}}_k = \mathcal{A}_k + \mathcal{A}_k^\dagger. \quad (19)$$

In addition, we have the following commutation relations

$$\begin{aligned} \mathbb{P}^\times(\mathcal{A}_k) &= [\mathbb{P}_k, \mathcal{A}_k] = ic_k^2 \mathbb{P} - \mathcal{A}_k, \\ \mathbb{P}^\times(\mathcal{A}_k^\dagger) &= [\mathbb{P}_k, \mathcal{A}_k^\dagger] = ic_k^2 \mathbb{P} + \mathcal{A}_k^\dagger, \end{aligned} \quad (20)$$

Intriguingly, we observe that the recursive structure of the nested-commutator appears

$$\begin{aligned} [\mathbb{P}_k, [\mathbb{P}_k, \dot{\mathbb{P}}_k]] &= [\mathbb{P}_k, [\mathbb{P}_k, \mathcal{A}_k + \mathcal{A}_k^\dagger]] \\ &= -[\mathbb{P}_k, \mathcal{A}_k] + [\mathbb{P}_k, \mathcal{A}_k^\dagger] \\ &= \mathcal{A}_k + \mathcal{A}_k^\dagger = \dot{\mathbb{P}}_k. \end{aligned} \quad (21)$$

With the help of the Taylor series expansion of the hyperbolic tangent function, the function f can be rewritten as

$$f(t) = \sum_{n=0}^{\infty} \frac{4(4^{n+1} - 1)B_{2n+2}}{(2n+2)!} t^{2n}, \quad (22)$$

where B_{2n+2} is the $(2n+2)$ th Bernoulli number. It is remarkable that $f(t)$ only consists of the even-order terms, which is compatible with the Hermiticity of the SLD.

Therefore, in our case, it is equivalent to define $G = \alpha\mathbb{P}(\phi) + \beta$. From the formula (16) and the commutation relation (21), we obtain the desired expression of the SLD

$$\mathcal{L}_k = \alpha f(\alpha) \dot{\mathbb{P}}_k = 2 \tanh(\alpha/2) \dot{\mathbb{P}}_k, \quad (23)$$

From Eq. (12), we finally get

$$\mathcal{L}_k = 2 \frac{e^\alpha - 1}{e^\alpha + 1} \dot{\mathbb{P}}_k = \frac{2d\eta}{2 + (d-2)\eta} \dot{\mathbb{P}}_k. \quad (24)$$

Note that it is easy to check that $\text{Tr}(\rho^w \mathcal{L}_k) = 0$, since $\text{Tr}(\mathbb{P} \dot{\mathbb{P}}_k) = 0$ and $\text{Tr}(\dot{\mathbb{P}}_k) = 0$.

Evaluation of QFIM. To compare with previous studies, here we analytically evaluate the QFIM of $\rho^w(\phi)$ by use of \mathcal{L}_k . In the multi-parameter scenario, the element of QFIM $\mathcal{F}(\phi) = [\mathcal{F}_{jk}]$ is defined by

$$\mathcal{F}_{jk} = \text{Tr} \left[\rho(\phi) \frac{\mathcal{L}_j \mathcal{L}_k + \mathcal{L}_k \mathcal{L}_j}{2} \right], \quad (25)$$

where \mathcal{L}_j and \mathcal{L}_k are SLDs with respect to ϕ_j and ϕ_k respectively. From Eq. (19), we have

$$\begin{aligned} \dot{\mathbb{P}}_k^2 &= c_k^2 (|k\rangle \langle k| + |\Psi\rangle \langle \Psi|) \\ &\quad - c_k^3 (e^{i\phi_k} |k\rangle \langle \Psi| + e^{-i\phi_k} |\Psi\rangle \langle k|), \end{aligned} \quad (26)$$

Since $\rho^w(\phi)$ can be regarded as a mixture of $\mathbb{P}(\phi)$ and the identity operator \mathbb{I} , the diagonal elements of QFIM is given by

$$\begin{aligned} \mathcal{F}_{kk} &= \text{Tr}(\rho^w \mathcal{L}_k^2) \\ &= (c_k^2 - c_k^4) \left[x + \frac{2(1-\eta)}{d} \right] \left[\frac{2d\eta}{2 + (d-2)\eta} \right]^2 \\ &= \frac{4d\eta^2}{2 + (d-2)\eta} (c_k^2 - c_k^4). \end{aligned} \quad (27)$$

Correspondingly, the product $\dot{\mathbb{P}}_j \dot{\mathbb{P}}_k$ ($j \neq k$) takes a similar form

$$\begin{aligned} \dot{\mathbb{P}}_j \dot{\mathbb{P}}_k &= -c_j c_k^2 e^{i\phi_j} |j\rangle \langle \Psi| - c_j^2 c_k e^{-i\phi_k} |\Psi\rangle \langle k| \\ &\quad + c_j c_k e^{i(\phi_j - \phi_k)} |j\rangle \langle k|, \end{aligned} \quad (28)$$

Therefore, the off-diagonal elements of QFIM is given by

$$\begin{aligned} \mathcal{F}_{jk} &= \text{ReTr}(\rho^w \mathcal{L}_j \mathcal{L}_k) \\ &= -c_j^2 c_k^2 \left[x + \frac{2(1-\eta)}{d} \right] \left[\frac{2d\eta}{2 + (d-2)\eta} \right]^2 \\ &= -\frac{4d\eta^2}{2 + (d-2)\eta} c_j^2 c_k^2 \quad (j \neq k). \end{aligned} \quad (29)$$

Finally, the QFIM can be represented in a compact form

$$\mathcal{F}_{jk} = \frac{4d\eta^2}{2 + (d-2)\eta} (c_j^2 \delta_{jk} - c_j^2 c_k^2), \quad \forall j, k. \quad (30)$$

Before moving forward, some remarks are in order. First, for the generalized d -dimensional equatorial pure states (e.g., $\eta = 1$ and $c_k = 1/\sqrt{d}$), we recover the result of Ref. [45]; meanwhile, if we only require that the amplitudes c_k are equal, the result in [16] is reestablished. Note that indeed the *Method II* is employed in Ref. [16], where a delicate choice of the decomposition basis plays a critical role in the analysis. On the other hand, as a result of the monotonicity of QFI [46], the following matrix inequality should be satisfied

$$\mathcal{F}(\rho^w(\phi)) \leq \eta \mathcal{F}(\mathbb{P}(\phi)), \quad (31)$$

where $\mathcal{F}(\mathbb{P}(\phi))$ denotes the QFIM of the pure state $|\Psi(\phi)\rangle \langle \Psi(\phi)|$. In fact, our result indicates that

$$\mathcal{F}(\rho^w(\phi)) = \frac{d\eta^2}{2 + (d-2)\eta} \mathcal{F}(\mathbb{P}(\phi)) \leq \eta \mathcal{F}(\mathbb{P}(\phi)). \quad (32)$$

Moreover, since the QFIM of $\rho^w(\phi)$ is proportional to that of $\mathbb{P}(\phi)$, we can define the *ratio* function

$$\xi(\eta) \doteq \frac{d\eta^2}{2 + (d-2)\eta} \leq 1. \quad (33)$$

It is easy to check that $\xi(\eta)$ is a monotonically increasing function of the shrinking factor η . This property of $\xi(\eta)$ confirms that (i) the QFI can *never* be amplified in the presence of white noise; (ii) the larger η is, the more

information $\rho^w(\phi)$ contains about parameters, which is to be expected.

Attainability of QCRB. In the multi-parameter scenarios, the celebrated quantum Cramér-Rao bound (QCRB) refers to the matrix inequality [17]

$$\text{Cov}(\phi) \geq [M\mathcal{F}(\phi)]^{-1}, \quad (34)$$

where $\text{Cov}(\phi)$ stands for the covariance matrix of the unbiased estimator $\hat{\phi}$ and M is the number of measurements repeated ($M = 1$ for definiteness). In sharp contrast to the single-parameter case, this lower bound cannot be achieved in general, since simultaneous estimation of multiple parameters usually involves the joint measurement of the corresponding *incompatible* observables. The attainability problem of QCRB for pure states has already been resolved by Fujiwara and Matsumoto [47, 48]. For general mixed states, only recently a series of research results by Guță *et al.* reveal that the QCRB is asymptotically attainable if and only if [49]

$$\text{Tr}(\rho(\phi)[\mathcal{L}_j, \mathcal{L}_k]) = 0 \Leftrightarrow \text{ImTr}(\rho(\phi)\mathcal{L}_j\mathcal{L}_k) = 0, \quad (35)$$

which is satisfied for all j and k . In our study, the above calculation clearly shows that

$$\text{Tr}(\rho^w(\phi)\mathcal{L}_j\mathcal{L}_k) = \text{ReTr}(\rho^w(\phi)\mathcal{L}_j\mathcal{L}_k) \in \mathbb{R}. \quad (36)$$

Therefore, the multi-parameter QCRB is achievable in this particular case. Taking the trace of both sides of QCRB, the total variance (error) of all the phases estimated follows the inequality

$$(\Delta\phi)^2 = \sum_{\mu=1}^{d-1} (\Delta\phi_\mu)^2 = \text{Tr}[\text{Cov}(\phi)] \geq \text{Tr}[\mathcal{F}(\phi)^{-1}]. \quad (37)$$

Note that this lower bound is also achievable due to the saturation of QCRB. In our case, it can be given as

$$(\Delta\phi)_{\min}^2 = \sum_{\mu=1}^{d-1} F_\mu^{-1}, \quad (38)$$

where $\{F_\mu\}_{\mu=1}^{d-1}$ is the set of eigenvalues of $\mathcal{F}(\rho^w(\phi))$. Remember that $\mathcal{F}(\rho^w(\phi))$ is a $(d-1) \otimes (d-1)$ matrix. In addition, the possible symmetry of $|\Psi(\phi)\rangle$ may help us to access an analytical lower bound [16].

To elucidate the optimal (joint) measurement of all the parameters, we follow the idea of Marzolino and Braun [50] and *generalize* their method to our discussion. Based on the diagonalization of the inverse of QFIM, the QCRB (34) can be transformed into

$$\text{Cov}(\lambda) \geq \Lambda(\lambda) = \mathcal{Q}\mathcal{F}(\phi)^{-1}\mathcal{Q}^T, \quad (39)$$

where we define the column parameter vectors $\vec{\phi} = \{\phi_k\}_{k=1}^{d-1}$, $\vec{\lambda} = \{\lambda_k\}_{k=1}^{d-1}$ and $\vec{\lambda} = \mathcal{Q}\vec{\phi}$. Here \mathcal{Q} is the

orthogonal matrix that diagonalizes $\mathcal{F}(\phi)^{-1}$ and the diagonal matrix $\Lambda(\lambda) = \text{diag}\{F_{\lambda_1}^{-1}, F_{\lambda_2}^{-1}, \dots, F_{\lambda_{d-1}}^{-1}\}$. Since $\lambda_i = \mathcal{Q}_{ij}\phi_j$ (Einstein's summation convention) and \mathcal{Q} is independent of $\vec{\phi}$ in our analysis, we have the equation

$$\begin{aligned} \frac{\partial \rho}{\partial \phi_j} &= \frac{1}{2}(\mathcal{L}_{\phi_j}\rho + \rho\mathcal{L}_{\phi_j}) = \frac{\partial \rho}{\partial \lambda_i}\mathcal{Q}_{ij} \\ &= \frac{1}{2}(\mathcal{L}_{\lambda_i}\rho + \rho\mathcal{L}_{\lambda_i})\mathcal{Q}_{ij}. \end{aligned} \quad (40)$$

Therefore, we have the relation $\mathcal{L}_{\phi_j} = \mathcal{Q}_{ij}\mathcal{L}_{\lambda_i}$, or equivalently, $\mathcal{L}_{\vec{\phi}} = \mathcal{Q}^T\mathcal{L}_{\vec{\lambda}}$, where $\mathcal{L}_{\vec{\phi}}$ and $\mathcal{L}_{\vec{\lambda}}$ are the corresponding SLD vectors. Due to $\mathcal{Q}^T\mathcal{Q} = \mathbb{I}$, we finally arrive at $\mathcal{L}_{\vec{\lambda}} = \mathcal{Q}\mathcal{L}_{\vec{\phi}}$. According to quantum estimation theory [3], the optimal quantum estimator vector $\mathcal{O}_{\vec{\lambda}}$ is given by

$$\mathcal{O}_{\lambda_k} = \lambda_k\mathbb{I} + \frac{\mathcal{L}_{\lambda_k}}{F_{\lambda_k}}, \quad (1 \leq k \leq d-1) \quad (41)$$

which attains the QCRB (39) and achieves the desired property

$$\text{Cov}(\mathcal{O}_{\lambda_j}, \mathcal{O}_{\lambda_k}) = 0, \quad (j \neq k) \quad (42)$$

implying that the optimal estimators for *distinct* parameters are uncorrelated and the joint measurement can be realized.

In summary, we investigate the multiple phase estimation problem for a natural parametrization of arbitrary pure states under *white noise*. Our analysis extends and unifies several partial results for more specified states [16, 45]. We have obtained the analytical and compact expression of QFIM and also confirmed that the QCRB is universally attainable in this scenario. Since the QFIM is irrespective of the parameters to be estimated, the lower bound of the total estimated error is proved to be a parameter-independent quantity, which might be significant in other contexts. We also illustrate the optimal estimators attaining the QCRB for future experimental purpose. Moreover, it is worth pointing out that our approach can be generalized to other circumstances, such as

$$\sigma = \frac{\eta}{r}\tilde{\mathbb{P}}(\phi) + \frac{1-\eta}{d}\mathbb{I}_{d \times d}, \quad (43)$$

where the projection operator $\tilde{\mathbb{P}}(\phi)$ is of the Lüders type [51] and r is the rank of $\tilde{\mathbb{P}}(\phi)$.

Acknowledgments. This research is supported by the National Natural Science Foundation of China (Grants No. 11025527, No. 11121403, No. 10935010, No. 11074261, and No. 11247006), the National 973 program (Grants No. 2012CB921602, No. 2012CB922104, and No. 2014CB921403), and the China Postdoctoral Science Foundation (Grant No. 2014M550598).

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- [1] V. Giovannetti, S. Lloyd, and L. Maccone, *Science* **306**, 1330 (2004).
 - [2] V. Giovannetti, S. Lloyd, and L. Maccone, *Phys. Rev. Lett.* **96**, 010401 (2006).
 - [3] M. G. A. Pairs, *Int. J. Quantum Inform.* **7**, 125 (2009).
 - [4] V. Giovannetti, S. Lloyd, and L. Maccone, *Nat. Photon.* **5**, 222 (2011).
 - [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, 2000).
 - [6] A. Fujiwara and H. Imai, *J. Phys. A* **41**, 255304 (2008).
 - [7] Z. Ji, G. Wang, R. Duan, Y. Feng, and M. Ying, *IEEE Trans. Inf. Theory* **54**, 5172 (2008).
 - [8] S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, *Phys. Rev. A* **83**, 021804(R) (2011).
 - [9] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Nat. Phys.* **7**, 406 (2011).
 - [10] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, *Nat. Commun.* **3**, 1063 (2012).
 - [11] D. Hosler and P. Kok, *Phys. Rev. A* **88**, 052112 (2013).
 - [12] M. Ahmadi, D. E. Bruschi, N. Friis, C. Sabín, G. Adesso, and I. Fuentes, *Sci. Rep.* **4**, 4996 (2014).
 - [13] Y. Yao, X. Xiao, L. Ge, X. G. Wang, and C. P. Sun, *Phys. Rev. A* **89**, 042336 (2014).
 - [14] X.-M. Lu, Z. Sun, X. Wang, S. Luo, and C. H. Oh, *Phys. Rev. A* **87**, 050302(R) (2013).
 - [15] H. Song, S. Luo, N. Li, and L. Chang, *Phys. Rev. A* **88**, 042121 (2013).
 - [16] Y. Yao, L. Ge, X. Xiao, X. G. Wang, and C. P. Sun, *Phys. Rev. A* **90**, 022327 (2014).
 - [17] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
 - [18] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
 - [19] S. L. Braunstein and C. M. Caves, *Phys. Rev. Lett.* **72**, 3439 (1994).
 - [20] S. L. Braunstein, C. M. Caves, and G. J. Milburn, *Ann. Phys. (N.Y.)* **247**, 135 (1996).
 - [21] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, *Phys. Rev. Lett.* **98**, 090401 (2007).
 - [22] B. M. Escher, L. Davidovich, N. Zagury, and R. L. de Matos Filho, *Phys. Rev. Lett.* **109**, 190404 (2012).
 - [23] S. Alipour, M. Mehboudi, and A. T. Rezakhani, *Phys. Rev. Lett.* **112**, 120405 (2014).
 - [24] M. Zwierz and H. M. Wiseman, *Phys. Rev. A* **89**, 022107 (2014).
 - [25] S. Pang and T. A. Brun, *Phys. Rev. A* **90**, 022117 (2014).
 - [26] D. Bures, *Trans. Amer. Math. Soc.* **135**, 199 (1969).
 - [27] A. Uhlmann, *Rep. Math. Phys.* **9**, 273 (1976).
 - [28] M. Hübner, *Phys. Lett. A* **163**, 239 (1992).
 - [29] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori, *Phys. Rev. A* **87**, 022337 (2013).
 - [30] O. Pinel, P. Jian, N. Treps, C. Fabre, and D. Braun, *Phys. Rev. A* **88**, 040102(R) (2013).
 - [31] G. Adesso, *Phys. Rev. A* **90**, 022321 (2014).
 - [32] C. J. O'Loan, arXiv:1001.3971.
 - [33] J. Liu, X. Jing, and X. Wang, *Phys. Rev. A* **88**, 042316 (2013).
 - [34] Y. M. Zhang, X. W. Li, W. Yang, and G. R. Jin, *Phys. Rev. A* **88**, 043832 (2013).
 - [35] J. Liu, X. Jing, W. Jing, and X. Wang, *Commun. Theor. Phys.* **61**, 45 (2014).
 - [36] D. Collins, *Phys. Rev. A* **87**, 032301 (2013).
 - [37] A. Monras, arXiv:1303.3682.
 - [38] S. I. Knysh and G. A. Durkin, arXiv:1307.0470.
 - [39] Z. Jiang, *Phys. Rev. A* **89**, 032128 (2014).
 - [40] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989).
 - [41] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **81**, 5932 (1998).
 - [42] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, *Phys. Rev. Lett.* **83**, 1054 (1999).
 - [43] V. Bužek and H. Hillery, *Phys. Rev. Lett.* **81**, 5003 (1998).
 - [44] Y. Tanimura and R. Kubo, *J. Phys. Soc. Jpn.* **58**, 101 (1989).
 - [45] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, *Phys. Rev. Lett.* **111**, 070403 (2013).
 - [46] A. Fujiwara, *Phys. Rev. A* **63**, 042304 (2001).
 - [47] A. Fujiwara, *Phys. Rev. A* **65**, 012316 (2001).
 - [48] K. Matsumoto, *J. Phys. A* **35**, 3111 (2002).
 - [49] <http://www.heraeus2011.rhul.ac.uk/Presentations/Guta.pdf>.
 - [50] U. Marzolino and D. Braun, *Phys. Rev. A* **88**, 063609 (2013).
 - [51] G. Lüders, *Ann. Phys.* **8**, 322 (1951).